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Optimization of space-time material layout for 1D wave propagation with varying mass and stiffness parameters

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Abstract

Results are presented for optimal layout of materials in the spatial and temporal domains for a 1D structure subjected to transient wave propagation. A general optimization procedure is outlined including derivation of design sensitivities for the case when the mass density and stiffness vary in time. The outlined optimization procedure is exemplified on a 1D wave propagation problem in which a single gaussian pulse is compressed when propagating through the optimized structure. Special emphasis is put on the use of a time-discontinuous Galerkin integration scheme that facilitates analysis of a system with a time-varying mass matrix.

Keywords: dynamic structures, topology optimization, wave propagation, transient analysis.

1 Introduction

The method of topology optimization is a popular method for obtaining the optimal layout of one or several material constituents in structures and materials (Bendsøe and Kikuchi, 1988; Bendsøe and Sigmund, 2003). The methodology has within the last two decades evolved into a mature and diverse research field involving advanced numerical procedures and various application areas such as fluids (Borrvall and Petersson, 2003), waves (Sigmund and Jensen, 2003), electromagnetism (Cox and Dobson, 1999) as well as various coupled problems such as e.g. fluid-structure interaction (Yoon et al., 2007). Additionally, industrial applications in the automotive and aerospace industries are established and widespread. The success has been facilitated by the large design freedom inherently associated with the concept, but also by efficient numerical techniques such as adjoint sensitivity analysis for rapid computation of gradients (Tortorelli and Michaleris, 1994), various penalization and regularization techniques for obtaining both meaningful and useful designs (Sigmund and Petersson, 1998) and the close integration with mathematical programming tools, such as the method of moving asymptotes (MMA) (Svanberg, 1987).

Recently it was suggested to apply (and somewhat extend) the standard topology optimization framework to design a 1D structure in which the stiffness could change both in space and time (Jensen, 2009). As an example an optimized "dynamic structure" that prohibits wave propagation was designed and manifested itself as a moving bandgap structure with layers of stiff inclusions moving with the propagating wave. The dynamic structure was demonstrated to reduce the transmission of a wave pulse with about a factor 3 compared to an optimized static structure. The present paper extends the described work by allowing materials that have not only time-varying stiffness but also a time-varying mass density. This extension requires special attention to the choice of time-integration scheme since many standard schemes fail. However, it allows for extended manipulation of the wave propagation as illustrated in the example in the present paper, in which a single gaussian wave pulse is compressed when propagating through the optimized structure. Preliminary results for this design problem in the case of time-varying stiffness were presented in (Jensen, 2008).

The basic setting for obtaining optimal space and time distributions of materials for problems governed by the wave equation was first presented in (Maestre et al., 2007; Maestre and Pedregal, 2009). These papers analyze 1D and 2D problems with a strong focus on the mathematical aspects of the optimization problem. Both the present paper and the aforementioned works root in the fundamental concept of dynamic materials. This concept was introduced by Lurie and Blekhman in (Lurie, 1997; Blekhman and Lurie, 2000; Lurie, 2006; Blekhman, 2008) who unfolded the rich and complex behavior of materials with properties that vary in space and time. At a similar time the dynamics of structures with space and time varying properties was also studied in the work by Krylov and Sorokin (Krylov and Sorokin, 1997) and later in (Sorokin et al., 2000; Sorokin and Grishina, 2004).

The basis for the presented optimization problem is time-integration of the transient model equation coupled with adjoint sensitivity analysis. Thus, the problem closely resembles previous studies that have been carried out for topology optimization of static structures using a transient formulation, e.g. (Min et al., 1999; Turteltaub, 2005; Dahl et al., 2008).

The outline of the paper is as follows. In Section 2 the governing equation is presented and the basic setup defined. In Section 3 the design parametrization is defined and design sensitivities are derived. Section 4 is devoted to numerical analysis of the transient direct and adjoint equations and numerical simulation results are presented. In Section 5 an optimization problem is defined and examples of optimized designs are presented. Section 6 summarizes and gives conclusions.

2 Governing equation

The starting point for the analysis and subsequent optimization study is a time-dependent FE model in which the mass matrix ($\mathbf{M}(t)$) and the stiffness matrix ($\mathbf{K}(t)$) are allowed to vary in time:

$$\frac{\partial}{\partial t}\big(\mathbf{M}(t)\mathbf{v}\big) + \mathbf{C}\mathbf{v} + \mathbf{K}(t)\mathbf{u} = \mathbf{f}(t) \tag{1}$$

in which $\mathbf{C}$ is a constant damping matrix and $\mathbf{f}(t)$ is the transient load. The vector $\mathbf{u}(t)$ contains the unknown nodal displacements and the notation $\mathbf{v} = \partial\mathbf{u}/\partial t$ has been used to denote the unknown velocities. It is assumed that the mass matrix is diagonal, e.g. obtained by a standard lumping procedure. This will be of importance when choosing a proper time-integration routine but it should be emphasized that all formulas derived in the following hold also for the case of $\mathbf{M}$ being non-diagonal.

The governing equation is solved in the time domain with the trivial initial conditions:

$$\mathbf{u}(t) = \mathbf{v}(t) = \mathbf{0} \tag{2}$$

which imposes only limited loss of generality and facilitates the sensitivity analysis as shown later.

It should be noted that although the terms mass matrix/mass density and stiffness matrix/stiffness are used here and in the following presentation, the equations could just as well apply to an electromagnetic or an acoustic problem with proper renaming of involved parameters. However, the terminology from elasticity will be kept throughout this paper.

3 Parametrization and sensitivities

The density approach to topology optimization (Bendsøe, 1989) is adapted to the present problem. With this approach a single design variable x_e ("density") is assigned to each element is the FE model. As in (Jensen, 2009) this is expanded to the space-time case by defining a vector of continuous design variables:

$$\mathbf{x}_j = \{x_j^1,\ x_j^2,\ \ldots,\ x_j^N\}^T \tag{3}$$

for each of a predefined number M of time intervals (such that $j \in [1, M]$), for which the design will be allowed to change. In Eq. (3) N is the number of spatial elements in the FE model. Thus, for a 1D spatial structure, as considered in the example in Section 5, the corresponding design space is two-dimensional with dimension $N \times M$.

The value of the density variable x_j^e will determine the material properties of that space-time element by an interpolation between two predefined materials 1 and 2, where the variable is allowed to take any value from 0 to 1 ($x_j^e \in [0; 1]$).

By rescaling the equations with respect to the material properties of material 1, the mass and stiffness matrices can be written as:

$$\mathbf{M}_j = \sum_{e=1}^{N} (1 + x_j^e(\rho - 1))\mathbf{M}^e \tag{4}$$

$$\mathbf{K}_j = \sum_{e=1}^{N} (1 + x_j^e(E - 1))\mathbf{K}^e \tag{5}$$

such that ρ, E denote the *contrast* between the two materials for the mass density and stiffness, respectively. In Eqs. (4)–(5), $\mathbf{M}^e$ and $\mathbf{K}^e$ are local mass and stiffness matrices expressed in global coordinates.

Analytical expressions for the design sensitivities are now derived. The optimization is based on an objective that is assumed to be written as:

$$\phi = \int_0^{\mathcal{T}} c(\mathbf{u})dt \tag{6}$$

in which c is a real scalar function of the time-dependent displacement vector and $\mathcal{T}$ is the total simulation time. It should emphasized that more complicated objective functions, e.g. with a dependence on the velocities or an integration different from the total simulation time, can be treated with minor modification to the following derivation.

The derivative wrt. a single design variable in the j'th time-interval and e'th spatial variable is denoted $()' = \partial/\partial x_j^e$ and thus the sensitivity of ϕ wrt. to x_j^e is:

$$\phi' = \int_0^{\mathcal{T}} \frac{\partial c}{\partial \mathbf{u}} \mathbf{u}' dt \tag{7}$$

Eq. (7) involves the term $\mathbf{u}'$ which is difficult to evaluate explicitly. However, the adjoint method can be used to circumvent this problem in an efficient way (Arora and Holtz, 1997). For this purpose the residual vector $\mathbf{R}$:

$$\mathbf{R} = \frac{\partial}{\partial t}(\mathbf{M}\mathbf{v}) + \mathbf{C}\mathbf{v} + \mathbf{K}\mathbf{u} - \mathbf{f}(t) \tag{8}$$

is differentiated wrt. x_j^e:

$$\mathbf{R}' = \frac{\partial}{\partial t}(\mathbf{M}'\mathbf{v} + \mathbf{M}\mathbf{v}') + \mathbf{C}\mathbf{v}' + \mathbf{K}'\mathbf{u} + \mathbf{K}\mathbf{u}' \tag{9}$$

in which it has been used that $\mathbf{f}$ (the transient load) and $\mathbf{C}$ (the damping matrix) are both independent of the design.

With the aid of Eq. (9), Eq. (7) is reformulated as:

$$\phi' = \int_0^{\mathcal{T}} (\frac{\partial c}{\partial \mathbf{u}} \mathbf{u}' + \boldsymbol{\lambda}^T \mathbf{R}')dt \tag{10}$$

in which $\boldsymbol{\lambda}$ denote an unknown vector of Lagrangian multipliers to be determined in the following.

Expanding the expression in Eq. (10) and using integration by parts leads to the following equation:

$$\begin{aligned}\phi' = \int_0^{\mathcal{T}} (\boldsymbol{\lambda}^T \mathbf{K}'\mathbf{u} - \boldsymbol{\gamma}^T \mathbf{M}'\mathbf{v})dt + \int_0^{\mathcal{T}} (\frac{\partial c}{\partial \mathbf{u}} + \frac{\partial}{\partial t}(\boldsymbol{\gamma}^T \mathbf{M}) - \boldsymbol{\gamma}^T \mathbf{C} + \boldsymbol{\lambda}^T \mathbf{K})\mathbf{u}' dt \\ + \left[\boldsymbol{\lambda}^T (\mathbf{M}'\mathbf{v} + \mathbf{M}\mathbf{v}' + \mathbf{C}\mathbf{u}') - \boldsymbol{\gamma}^T \mathbf{M}\mathbf{u}'\right]_0^{\mathcal{T}} \end{aligned} \tag{11}$$

in which the notation $\boldsymbol{\gamma} = \partial\boldsymbol{\lambda}/\partial t$ has been introduced.

Now the unknowns $(\boldsymbol{\lambda}, \boldsymbol{\gamma})$ can be chosen so that the last integral in expression (11) vanishes along with the bracketed term that originates in the boundary contribution from integrating by parts (if the trivial initial conditions in Eq. (2) are applied as well). This leads to the following adjoint equation:

$$\frac{\partial}{\partial t}(\mathbf{M}^T \boldsymbol{\gamma}) - \mathbf{C}^T \boldsymbol{\gamma} + \mathbf{K}^T \boldsymbol{\lambda} = -(\frac{\partial c}{\partial \mathbf{u}})^T \tag{12}$$

along with the following terminal conditions:

$$\boldsymbol{\lambda}(\mathcal{T}) = \boldsymbol{\gamma}(\mathcal{T}) = \mathbf{0} \tag{13}$$

The sensitivities can then be computed from the remaining expression:

$$\phi' = \int_0^{\mathcal{T}} (\boldsymbol{\lambda}^T \mathbf{K}'\mathbf{u} - \boldsymbol{\gamma}^T \mathbf{M}'\mathbf{v})dt = \int_{\mathcal{T}_j^-}^{\mathcal{T}_j^+} (\boldsymbol{\lambda}^T \mathbf{K}'\mathbf{u} - \boldsymbol{\gamma}^T \mathbf{M}'\mathbf{v})dt \tag{14}$$

in which the integral can be reduced to the j'th time interval ranging from $\mathcal{T}_j^-$ to $\mathcal{T}_j^+$ simply because $\mathbf{K}'$ and $\mathbf{M}'$ vanish outside the interval belonging to the specific design variable.

The expression can be further reduced to element level as follows:

$$\phi' = \int_{\mathcal{T}_j^-}^{\mathcal{T}_j^+} \big((E-1)(\boldsymbol{\lambda}^e)^T \mathbf{K}^e \mathbf{u}^e - (\rho - 1)(\boldsymbol{\gamma}^e)^T \mathbf{M}^e \mathbf{v}^e\big)dt \tag{15}$$

by using the material interpolations defined in Eqs. (4)–(5).

4 Numerical analysis

Special care has to be taken to solve the direct and adjoint problems in Eqs. (1)–(2) and Eqs. (12)–(13) in the case where $\mathbf{M}$ is not constant. In this case, $\mathbf{v}$ and $\boldsymbol{\gamma}$ are not continuous and the numerical integration scheme must be able to handle this difficulty. A time-discontinuous Galerkin procedure (Wiberg and Li, 1999) allows for discontinuous field variables in time domain and is applicable for this case. An explicit version of the scheme is applied. The choice of an explicit solver (in combination with a lumped mass matrix) is essential for an efficient solution of the equations.

The basic numerical procedure is described shortly in the following for the direct problem of solving for $\mathbf{u}(t), \mathbf{v}(t)$. The adjoint problem for $\boldsymbol{\lambda}(t), \boldsymbol{\gamma}(t)$ is solved in a similar way. The total simulation time $\mathcal{T}$ is divided into N_t equidistant intervals and a discrete set of displacement and velocity vectors $\mathbf{u}_i, \mathbf{v}_i$ is obtained for $i \in [1, N_t + 1]$ including the initial conditions. Each time interval ($k \in [1, N_t]$) is treated as a time element and an inner-loop iterative procedure is used to obtain a velocity vector at the beginning of the interval denoted $\mathbf{v}_1^k$ and one at the end of the interval denoted $\mathbf{v}_2^k$. For the n'th inner-loop iteration the updates of $\mathbf{v}_1^k$ and $\mathbf{v}_2^k$ are:

$$\begin{aligned} \mathbf{M}(\mathbf{v}_1^k)^n &= (\mathbf{M}\mathbf{v}_2)^{k-1} + \tfrac{\Delta t}{6}(\mathbf{f}_1 - \mathbf{f}_2) \\ &+ \tfrac{(\Delta t)^2}{18}\mathbf{K}(\mathbf{v}_1^k - 2\mathbf{v}_2^k)^{n-1} - \tfrac{\Delta t}{6}\mathbf{C}(\mathbf{v}_1^k - \mathbf{v}_2^k)^{n-1} \end{aligned} \tag{16}$$

$$\begin{aligned} \mathbf{M}(\mathbf{v}_2^k)^n &= ((\mathbf{M}\mathbf{v}_2)^{k-1} - \Delta t(\mathbf{K}\mathbf{u})^{k-1}) + \tfrac{\Delta t}{2}(\mathbf{f}_1 + \mathbf{f}_2) \\ &- \tfrac{(\Delta t)^2}{6}\mathbf{K}(2\mathbf{v}_1^k - \mathbf{v}_2^k)^{n-1} - \tfrac{\Delta t}{2}\mathbf{C}(\mathbf{v}_1^k + \mathbf{v}_2^k)^{n-1} \end{aligned} \tag{17}$$

in which $\mathbf{f}_1$ and $\mathbf{f}_2$ is the load vector evaluated at the beginning and end of the time interval, respectively. The values of $(\mathbf{v}_1^k)^{n-1}$ and $(\mathbf{v}_2^k)^{n-1}$ for the initial iteration ($n = 1$) are taken to be equal to the value of $\mathbf{v}_2^{k-1}$. These inner loop iterations are continued until $\mathbf{v}_1^k$ and $\mathbf{v}_2^k$ do not change more than some predefined small tolerance (usually 2-3 iterations are performed).

Based on the converged time element values the recorded velocity and displacement vector at discrete time i is then:

$$\mathbf{v}^i = \mathbf{v}_2^k \tag{18}$$

$$\mathbf{u}^i = \mathbf{u}^{i-1} + \tfrac{\Delta t}{2}(\mathbf{v}_1^k + \mathbf{v}_2^k) \tag{19}$$

4.1 Test problem

The explicit time-discontinuous Galerkin formulation is now compared to a standard explicit central difference scheme as previously employed in (Jensen, 2009). The model problem is depicted in Fig. 1 and the setting is described in the following. A sine-modulated gaussian pulse propagates in a homogeneous medium with material properties $\rho = E = 1$ and a $t = t_0$ the material properties change instantaneously to $\rho = \rho_0$ and $E = E_0$. As a result the propagating wave splits up into a forward and a backward travelling wave. It can be shown analytically that the relative change in wave energy at the moment of change of material properties is given as:

$$\frac{\Delta E}{E} = \tfrac{1}{2}(E_0 + \frac{1}{\rho_0}) - 1 \tag{20}$$

In Fig. 2 the wave energy is plotted as a function of time. Both plots in the figure correspond to the case where the material properties are changed at $t_0 = 0.85\,\mathrm{s}$. In the first plot the material properties are $\rho_0 = 1$ and $E_0 = 1.5$, which correspond to a relative energy jump of 0.25 and as appears from the plot this jump is accurately predicted by the time-discontinuous Galerkin procedure but also with a normal central difference scheme. In the second plot $\rho_0 = 2$ and $E = 1.5$ are chosen and thus zero energy jump should occur. From the plot we can see that the time-discontinuous scheme correctly captures the behavior as opposed to the central difference scheme.

It should be mentioned that the time-discontinuous scheme is computationally more expensive than the straightforward central-difference scheme since it involves inner loop iterations. This depends on the specific value of the tolerance set for the inner-loop iterations (see(Wiberg and Li, 1999) for more details). It is possible that more efficient schemes could be developed.

5 Example: pulse compression

The optimization algorithm is now demonstrated on the particular design problem illustrated in Fig. 3. A single gaussian pulse is send through a one-dimensional structure and the transmitted wave is recorded. Wave propagation in the bar is simulated by applying a time-dependent load at the left boundary and adding absorbing boundary conditions in form of simple dampers in both ends.

The purpose of the optimization problem is to design the structure so that that the difference between the recorded output and a specified target is minimized. Thus, the following objective function is considered:

$$\phi = \int_0^{\mathcal{T}} (u_{\text{out}} - u^*_{\text{out}})^2 dt \tag{21}$$

in which u_{out} is the displacement history of the output point, u^*_{out} is the output point target, and $\mathcal{T}$ is the total simulation time.

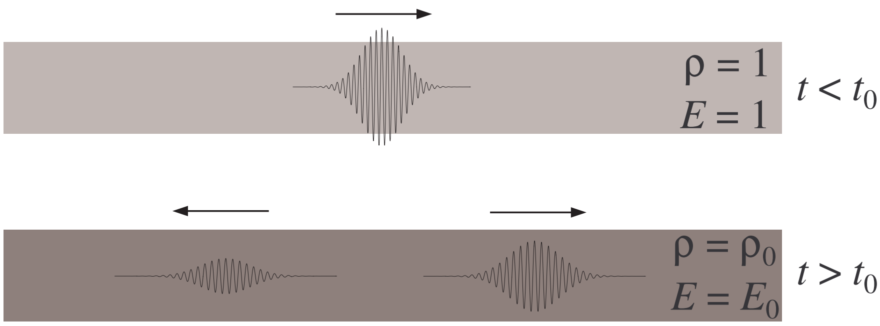
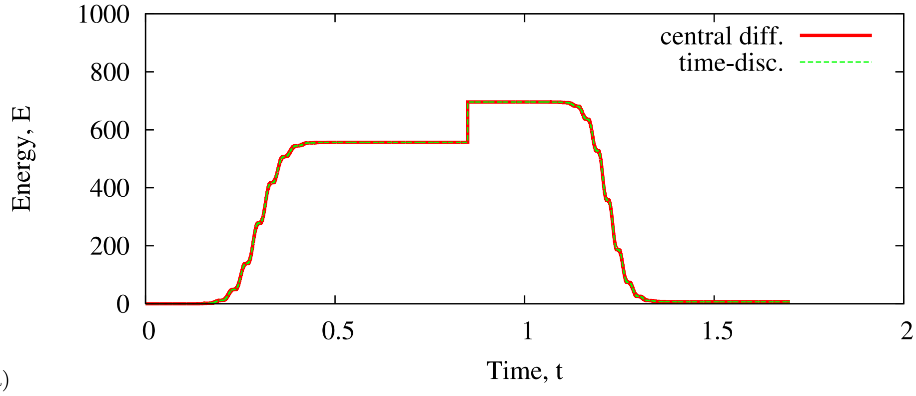


Figure 1: Propagation of a sine-modulated gaussian pulse in a homogeneous medium with an instant change of material properties at $t = t_0$.



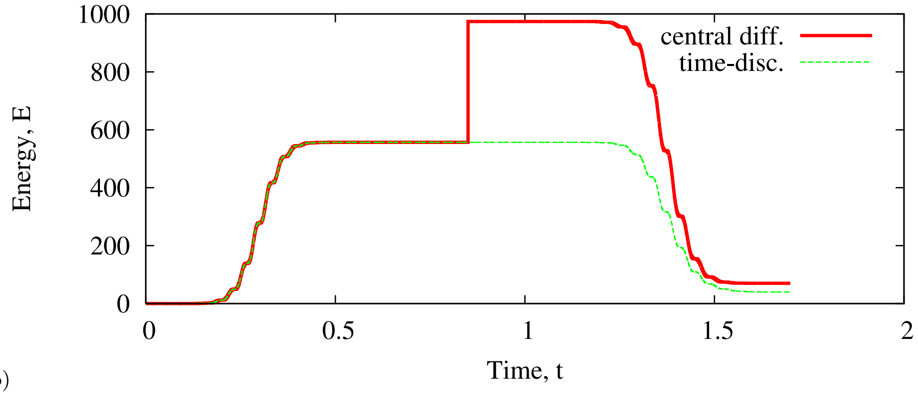


Figure 2: Simulation values of the wave energy in the homogeneous structure with an instantaneous change of material properties at $t_0 = 0.85\,\mathrm{s}$. a) $\rho_0 = 1$ and $E_0 = 1.5$ and b) $\rho_0 = 2$ and $E_0 = 1.5$.

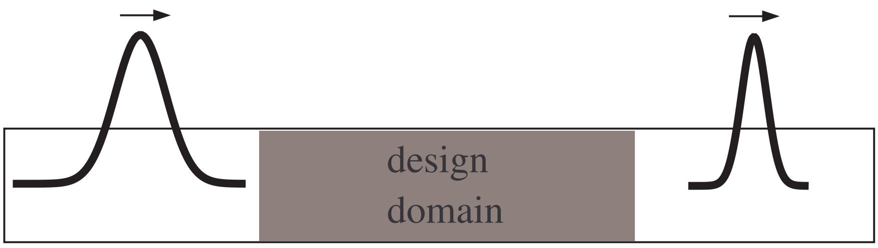


Figure 3: Design problem. A single gaussian pulse is to be compressed when propagating through the design domain by a suitable stiffness and mass density distribution in space and time.

The wave pulse is generated by applying the following force at the input point:

$$f(t) = -4u_0\delta(t-t_0)e^{-\delta(t-t_0)^2} \tag{22}$$

in which δ determines the width of the pulse, t_0 is the time center for the pulse, and u_0 is the amplitude of the resulting input wave pulse:

$$u(t) = u_0 e^{-\delta(t-t_0)^2} \tag{23}$$

As the target output pulse we choose

$$u^*_{\mathrm{out}}(t) = \tilde{u}_0 e^{-\tilde{c}\delta(t-\tilde{t}_0)^2} \tag{24}$$

in which $\tilde{c}$ represents the specified compression of the pulse.

5.1 Auxiliary design variables

In Eq. (24) the pulse time center at the output point is specified as $\tilde{t}_0$ and the amplitude of the output wave is specified to be $\tilde{u}_0$. Instead of fixing these values they are included in the optimization problem via extra design variables.

It is obvious that a reshaping of the wave leads to some delay of the pulse and the best value of $\tilde{t}_0$ is not known *a priori* and is thus natural to include in the design problem. The value of $\tilde{t}_0$ is given as:

$$\tilde{t}_0 = (\tilde{t}_0)_{\min} + x_1((\tilde{t}_0)_{\max} - (\tilde{t}_0)_{\min}) \tag{25}$$

so that the corresponding extra design variable x_1 takes values from 0 to 1. The minimum and maximum values are simply chosen large enough so that the value of x_1 does not reach the 0 or 1 limit during the optimization process.

The extra design variable x_2 associated with the output wave amplitude $\tilde{u}_0$ is defined as follows:

$$\tilde{u}_0 = (\tilde{u}_0)_{\min} + x_2((\tilde{u}_0)_{\max} - (\tilde{u}_0)_{\min}) \tag{26}$$

in which the minimum and maximum values are specified as values close to u_0, e.g. $(\tilde{u}_0)_{\min} = 0.8u_0$ and $(\tilde{u}_0)_{\max} = 1.2u_0$. In this way the optimization problem is relaxed somewhat in order to allow the optimization algorithm to find an optimal compression of the pulse without a too strict constraint on the pulse amplitude. It should be emphasized that obtaining an output pulse with an amplitude larger than the input pulse is possible also for an uncompressed pulse, since the energy is not conserved due to the external control of the material properties.

The sensitivities wrt. the auxiliary design variables can be obtained in a straightforward manner from Eqs. (21), (24)–(26).

5.2 Optimization problem

The optimization problem can now be written as:

$$
\begin{array}{rl}
\min_{\mathbf{x}_j, x_1, x_2} & \phi = \int_0^{\mathcal{T}} (u_{\text{out}} - u_{\text{out}}^*)^2 dt \\
\text{s.t.}: & \frac{\partial}{\partial t}\big(\mathbf{M}(t)\mathbf{v}\big) + \mathbf{C}\mathbf{v} + \mathbf{K}(t)\mathbf{u} = \mathbf{f}(t) \\
& t \in [0; \mathcal{T}] \\
& \mathbf{u}(0) = \mathbf{v}(0) = \mathbf{0} \\
& 0 \leq \mathbf{x}_j \leq 1, \quad j \in [1, M] \\
& 0 \leq x_1 \leq 1 \\
& 0 \leq x_2 \leq 1
\end{array}
\tag{27}
$$

and is solved using the derived expressions for the design sensitivities in combination with the method of moving asymptotes (Svanberg, 1987).

5.3 Results and discussion

In the following results are presented for the optimization problem described above. The model and simulation details are as follows. A unit length design domain is split into $N = 500$ spatial elements. The total simulation time is chosen to be $\mathcal{T} = 1.8\,\text{s}$ and the numerical time-integration is performed using $N_t = 9000$ time steps. The input pulse is defined via the parameters $u_0 = 1$, $\delta = 100\,\text{s}^{-2}$ and $t_0 = 0.3\,\text{s}$.

The optimization problem is defined by specifying the target pulse with a compression corresponding to $\tilde{c} = 3.5$. The limits for the auxiliary design variables are chosen to be $(\tilde{t}_0)_{\min} = 1.2\,\text{s}$, $(\tilde{t}_0)_{\max} = 1.35\,\text{s}$, $(\tilde{u}_0)_{\min} = 0.8u_0$ and $(\tilde{u}_0)_{\max} = 1.2u_0$. The design is allowed to change $M = 36$ times during the simulation time and in order to keep the designs simpler the spatial elements are grouped into 20 patches. Thus, the total number of design variables in the model becomes $20 \times 36 + 2 = 722$.

Fig. 4 shows an example of a pulse that is compressed when propagating through a space-time optimized structure obtained with material parameters $\rho = 1$ and $E = 1.75$. The curves in Fig. 4 additionally illustrate how the pulse, apart from being compressed, is delayed in the optimized structure when compared to the pulse propagating in the homogeneous structure. In this case the optimized value of the delay parameter is $\tilde{t}_0 \approx 1.25\,\text{s}$, whereas the optimized value of the output pulse amplitude $\tilde{u}_0$ is very close to the input pulse amplitude $u_0 = 1$.

In Fig. 5 four plots are presented, each showing a compressed output pulse compared to the target output pulse, each for a different set of material parameter contrasts ρ and E. Note, that the targets are different for the four plots since they depend on the optimized values of the auxiliary design variables x_1 and x_2.

Fig. 5a,b are obtained for structures that are optimized with a constant value of $\rho = 1$ but two different values of E (stiffness contrast). For low E ($E = 1.25$) it is evident that the target compression of the pulse cannot be obtained. There is a discrepancy between the curves near the tip and at the pulse front and tail where the pulse has not been compressed enough. However, when the contrast is

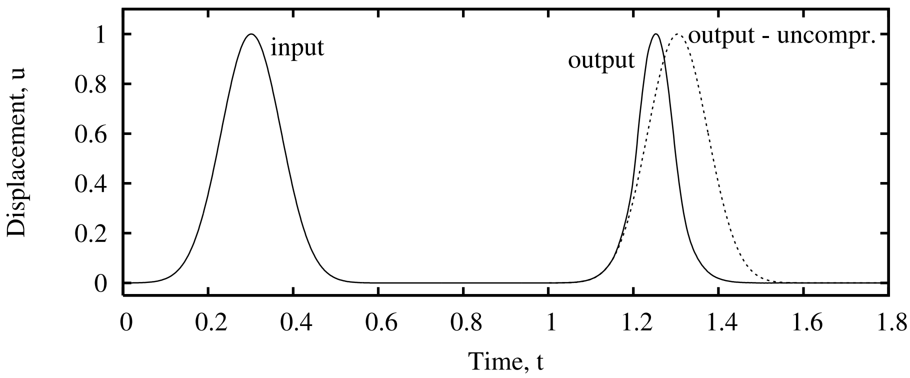


Figure 4: Left: input wave pulse. Right: optimized compressed wave pulse and for comparison the uncompressed output wave pulse.

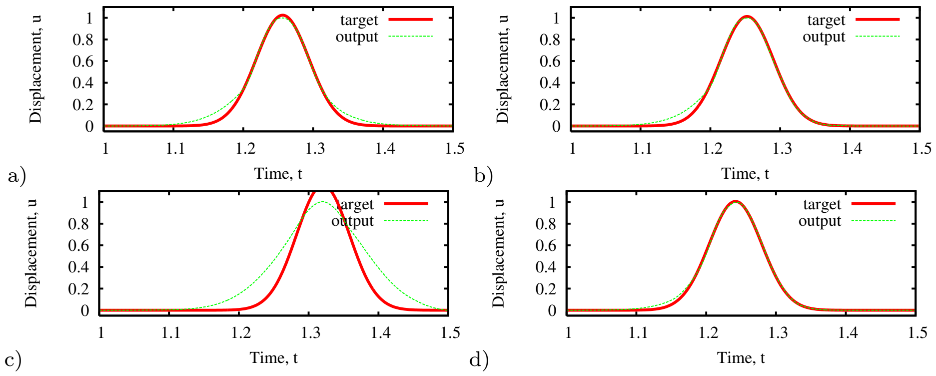


Figure 5: Four examples of output pulse through the optimized structure and the corresponding target pulse. The material parameters are: a) $\rho = 1$, $E = 1.25$, b) $\rho = 1$, $E = 1.75$, c) $\rho = 1.33$, $E = 1.25$, d) $\rho = 0.75$, $E = 1.25$.

increased ($E = 1.75$) a much better match to the target is obtained. The pulse front (corresponding to the part of the curve near $t = 1.1$ s) is still somewhat off the target.

In Fig. 5c,d the stiffness contrast is kept at the lower value ($E = 1.25$), but now the mass density contrast is changed to $\rho = 1.33$ and $\rho = 0.75$, respectively. It is evident from Fig. 5c that for this material property combination ($\rho = 1.33$) the targeted pulse compression is not possible at all, whereas for $\rho = 0.75$ the compression of the pulse is nearly perfect (with the pulse front still being slightly off). Thus it is clear that the combined effect of the two material parameters is very important and they should be chosen carefully in order to obtain the desired compression effect.

In the examples shown, the corresponding design variables range broadly from 0 to 1 (see Fig. 6a) which implies that the corresponding material properties in

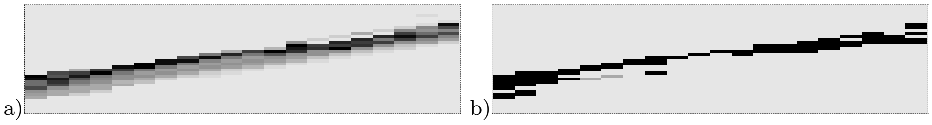


Figure 6: Space-time plot of the material distribution in the optimized structure. a) without penalization of intermediate densities, b) with penalization of intermediate densities.

the structure should be interpolations of material 1 and material 2. There is nothing in the optimization formulation as stated, that forces a binary 0-1 design that would be creatable with only the two materials available. If it is required that the structure can be fabricated with only the two specified sets of material properties, an explicit penalization scheme can be employed (e.g. (Borrvall and Petersson, 2001)). In this way the objective is appended with a penalizing term:

$$\phi = \int_0^{\mathcal{T}} (u_{\text{out}} - u^*_{\text{out}})^2 dt + \epsilon \sum_{j=1}^{M} \sum_{e=1}^{N} x_j^e (1 - x_j^e) \tag{28}$$

and in this way intermediate values of the design variables (between 0 and 1) are expensive and the design will inevitably be pushed toward a binary 0-1 design if the parameter ϵ is sufficiently large.

In Fig. 6a the space-time design variables in the optimized designs are plotted for the case of $\rho = 0.75$, $E = 1.25$ and in Fig. 6b the design variables are plotted with the optimization performed on the new objective function with explicit penalization in Eq. (28). The penalization has been employed by using the non-penalized structure as a staring point and increasing the value of ϵ in a number of steps using a continuation approach until most of the design variable take values that are 0 or 1. As appears from the figure only a few of the design variables are now intermediate. However, Fig. 7 shows that the almost perfect 0-1 design has been obtained at the cost of some of the performance of the structure. Especially, near the pulse tail the output pulse for the 0-1 optimized structure is quite different from the target.

Finally, in order to further illustrate the space-time distribution of the material properties, Fig. 8 show snapshots of the design variables along with the wave profile at four different time instances. The plots are for the non-penalized structure in Fig. 6a.

6 Summary and conclusions

This paper reports on a topology optimization procedure for the distribution of material in space and time. The procedure is applied to a 1D transient wave propagation problem in which a gaussian wave pulse is compressed when propagating through the structure which is composed of materials with different mass and stiffness parameters.

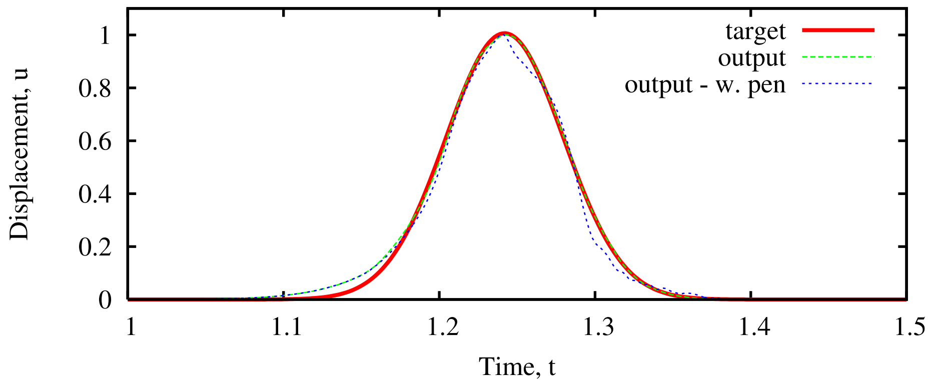


Figure 7: Target output pulse and output pulse for the optimized design without and with penalization of intermediate design variables.

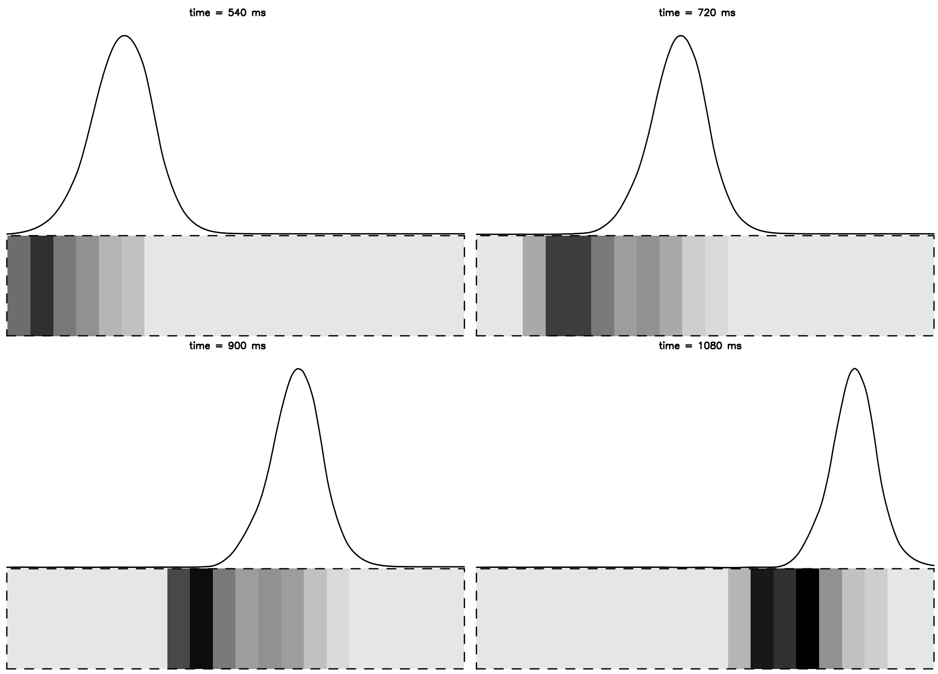


Figure 8: Illustration of the compression of the pulse as it propagates through the dynamic structure, corresponding to the space-time design shown in Fig. 6a and the output pulse in Fig. 5d.

Expressions for the design sensitivities are derived using the adjoint method. This leads to a terminal value transient problem. The direct and the adjoint discretized equations are solved using a time-discontinuous Galerkin procedure. This allows for correct simulation of the system when the mass matrix is not constant in time, however, at the expense of extra computational effort. The performance of the numerical scheme is demonstrated on a wave propagation problem in which the material properties change instantly. It is shown that the time-discontinuous scheme correctly simulates the problem whereas a standard central difference scheme fails if the mass matrix is not constant in time.

The optimization procedure is demonstrated on a 1D wave propagation problem in which a single gaussian pulse is compressed through an optimized space and time distribution of two materials with different mass density and stiffness. The optimization problem is formulated as a minimization problem in which the difference between the output pulse and a specified target output is minimized. Two auxiliary design parameters are introduced to relax the problem. They control the temporal location of the output pulse and its amplitude, which are allowed to vary within some predefined limits. The optimization problem is solved with the mathematical programming tool MMA.

It is shown that is it possible to compress the pulse depending on the specific values of mass and stiffness contrasts but that the designs will be composed of material properties that are mixtures of two predefined materials. An explicit penalization scheme is finally introduced in order to eliminate intermediate design variables so that the designs are primarily composed of the two available materials. This is shown to compromise the performance to some extend. The example clearly demonstrates that the pulse compression can be accomplished by using the presented scheme and indicates promising perspectives for using space-time topology optimization to create devices for more complex pulse shaping.

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